A SEQUENCE OF DISCRETE MINIMAL ENERGY CONFIGURATIONS THAT DOES NOT CONVERGE IN THE WEAK-STAR TOPOLOGY

MATTHEW T. CALEF

Abstract. We demonstrate a set A and a value of s for which the sequence of N-point discrete minimal Riesz s-energy configurations on A does not have an asymptotic distribution in the weak-star sense as N tends to infinity.

1. Introduction

In the study of classical electrostatics the representation of a large collection of electrons on a conductor by a charge density function is commonplace. In many cases such a representation is both physically and mathematically grounded. Here we consider an interaction potential derived from electrostatics and certain conductor geometries as a means to establish limits on when such a representation of many point charges is valid. In particular we demonstrate that, for certain fractal conductor geometries and Riesz potentials, there is no limiting charge distribution.

Consider a conductor A as a compact subset of $\mathbb{R}^p$ with Hausdorff dimension d. Let $\omega_N = \{x_1, \ldots, x_N\}$ denote the locations of N electrons on A. The electrostatic energy of ω_N is, up to a constant,

$$E_s(\omega_N) := \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{|x_i - x_j|^s},$$

where s is chosen to be one. By varying s the Riesz s-kernel $|x-y|^{-s}$ can represent generalizations of the Coulomb potential that decay or are singular to varying degrees.

The infimum of the N-point s-energy is denoted by

$$\mathcal{E}_s(A, N) := \inf_{\omega_N \subset A} E_s(\omega_N).$$

We extend $\mathcal{E}_s$ so that $\mathcal{E}_s(A, 0) = \mathcal{E}_s(A, 1) := 0$. For convenience we exclude the trivial case that A has finitely many points. For any positive value of s the functional E_s is lower semicontinuous, therefore, by the compactness of A, there is at least one configuration, which is denoted ω_N^s, that satisfies

$$\mathcal{E}_s(A, N) = E_s(\omega_N^s),$$

Identifying a minimal configuration ω_N^s for even simple conductor geometries such as $\mathbb{S}^2$ is a formidable task. Alternatively, one may study qualitative properties of ω_N^s as N tends to infinity. A common way to do this recasts the problem in terms of measures. For each N and s construct the probability measure

$$\mu^{s,N} = \frac{1}{N} \sum_{x \in \omega_N^s} \delta_x$$

that places a scaled Dirac-measure δ at each of the points in ω_N^s. One then considers measure-theoretic properties of the sequence of measures $\{\mu^{s,N}\}_{N=2}^{\infty}$.

In the case when $s < d$ one may formulate a continuous version of this problem as follows: Let $\mathcal{M}(A)$ denote the (unsigned) Borel measures supported on A. Let $\mathcal{M}_1(A) \subset \mathcal{M}(A)$ denote the Borel probability supported on A. If we represent a charge density by a measure $\mu \in \mathcal{M}(A)$ then the Riesz s-energy of μ is

$$I_s(\mu) := \iint \frac{1}{|x-y|^s} d\mu(y) d\mu(x),$$

Los Alamos National Laboratory, an affirmative action/equal opportunity employer, is operated by Los Alamos National Security, LLC, for the National Nuclear Security Administration of the U.S. Department of Energy under contract DE-AC52-06NA25396.

and the electrostatic ($s = 1$) energy of μ is $I_1(\mu)$. We may then consider the minimization problem

$$\mu^s = \arg\min\{I_s(\mu) \,:\, \mu \in \mathcal{M}_1(A)\}.$$

It is well known (cf [8]) that μ^s exists and is unique, that

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^2} = I_s(\mu^s),$$

and that for any function f that is continuous on A

$$\lim_{N\to\infty} \int f\, d\mu^{s,N} = \int f\, d\mu^s.$$

The last condition is referred to as weak-star convergence of $\mu^{s,N}$ to μ^s. A natural interpretation is that μ^s is the continuous equilibrium charge distribution on A and that it is a limit, as N tends to infinity, of minimal s-energy N-point configurations. These results have practical value as it is often substantially easier to obtain finite-dimensional approximations of μ^s than it is to determine the points that make up ω_N^s for large N.

In the case when $s \geq d$ every non-zero Borel measure supported on A has infinite s-energy (cf. [9]) and there is no obvious candidate for a continuous charge distribution that is the limit of the discrete minimal energy configurations. For this range of s recent results in [7, 3, 2] show that when A has certain rectifiability properties, the N-point minimal configurations are asymptotically uniform, i.e. $\{\mu^{s,N}\}_{N=2}^{\infty}$ converges in the weak-star sense to a suitably normalized d-dimensional Hausdorff measure $\mathcal{H}^d$ restricted to A. Additionally, the order of growth of $\mathcal{E}_s(A,N)$ is shown to be $N^{1+s/d}$ when $s > d$ and $N^2 \log N$ when $s = d$.

In [5] a candidate for a normalized $s = d$ energy is presented. (An alternate normalization for the case $s = d$ is presented by Gustafsson and Putinar in [6]. Putinar uses this normalization to study inverse moments problems in [11].) The work in [5, 4] was motivated by the hope that one could formulate, at least in the case $s = d$, a normalized energy that could be used to obtain results for the limit as N tends to infinity of the discrete minimal d-energy configurations. This effort was able to show that, for certain rectifiable or fractal sets, the normalized energy is uniquely minimized $\mathcal{H}^d$ restricted to A and normalized to have mass 1 – i.e. the uniform measure – and that μ^s converges in the weak-star sense to this uniform measure as s approaches d from below. However, it is not yet clear what can be inferred about the limit as N grows of the discrete minimal d-energy configurations.

Relatedly, the results presented in [7] rely heavily on characteristics – e.g. the local structure of the configurations ω_N^s – that are lost in the weak-star limit. Further, results from [2] show that for s sufficiently large and A in a class of self-similar fractals these local characteristics can cause the discrete minimal energy to oscillate to leading order as N tends to infinity. These results suggest that there are limits to what can be obtained from any continuous minimization problem, and this paper affirms this. By using the oscillations in energy demonstrated in [2], we demonstrate a value of s and a set A where any sequence of minimal N-point s-energy configurations does not converge in the weak-star sense. For such a set there can be no valid continuous representation of the limiting distribution as the number of points grows to infinity.

1.1. **Main Results.**

We consider, for some compact $K \subset \mathbb{R}^p$ of Hausdorff dimension d, the functions (cf. [7])

$$\underline{g}_{s,d}(K) = \liminf_{N\to\infty} \frac{\mathcal{E}_s(K,N)}{N^{1+s/d}} \qquad \text{and} \qquad \overline{g}_{s,d}(K) = \limsup_{N\to\infty} \frac{\mathcal{E}_s(K,N)}{N^{1+s/d}}.$$

If $\underline{g}_{s,d}(K) = \overline{g}_{s,d}(K)$ we denote the common value by $g_{s,d}(K)$. With this we prove the following theorem.

Theorem 1.1. *Let $A \subset \mathbb{R}^p$ be the disjoint union of two compact sets A_1 and A_2 of Hausdorff dimension $d > 0$ satisfying the following for some $s > d$:*

1. $\operatorname{diam}(A_1)$ *and* $\operatorname{diam}(A_2)$ *are both less than* $d(A_1, A_2) := \inf\{|x - y| : x \in A_1,\, y \in A_2\}$,
2. $0 < \underline{g}_{s,d}(A_1) < \overline{g}_{s,d}(A_2) < \infty$,
3. $g_{s,d}(A_2)$ *exists and is positive and finite.*

Then the sequence of measures $\{\mu^{s,N}\}_{N=2}^{\infty} \subset \mathcal{M}_1(A)$ cannot converge in the weak-star topology on $\mathcal{M}_1(A)$.

The central idea behind the proof of Theorem 1.1 is that, if $\mathcal{E}_s(A_1, N)$ oscillates to leading order, i.e. $N^{1+s/d}$, then the ratio of the number of points in $\omega_N^s \cap A_1$ to the number of points in $\omega_N^s \cap A_2$ cannot be constant. This is sufficient to show that if one chooses a Urysohn function ϕ that is 1 on A_1 and 0 on A_2, then the limit

$$\lim_{N\to\infty} \int \phi d\mu^{s,N}$$

cannot exist, which is enough to show that $\{\mu^{s,N}\}_{N=2}^{\infty}$ cannot have a weak-star limit.

The rest of this paper is organized as follows: Section 2 describes the class of sets that we consider and provides an example in this class based on the results in [2]. A particular weak-star cluster point as N tends to infinity for the discrete minimal N-point energy is demonstrated. Section 3 proves a lemma regarding the rate of change of the ratio of the minimal N-point energy divided by its leading order in N. Section 4 uses the results from the previous sections to show that, for the ranges of s and sets under consideration, there is no weak-star limit to the sequence $\{\mu^{s,N}\}_{N=2}^{\infty}$.

2. The Set A and a Weak Star Cluster Point

We shall consider a set A that is the disjoint union of two compact set A_1 and A_2. We require that $\mathrm{diam}(A_1)$ and $\mathrm{diam}(A_2)$ are both less than $d(A_1, A_2) = \inf\{|x-y| : x \in A_1, y \in A_2\}$. We shall further restrict ourselves to such sets satisfying $0 < \underline{g}_{s,d}(A_1) < \overline{g}_{s,d}(A_1) < \infty$ and $0 < g_{s,d}(A_2) < \infty$.

2.1. **An example of such a set.**

In [7] Hardin and Saff show that if A_2 is a compact subset of a d-dimensional C^1-manifold embedded in $\mathbb{R}^p$, satisfying $0 < \mathcal{H}^d(A_2) < \infty$, then $0 < \underline{g}_{s,d}(A_2) = \overline{g}_{s,d}(A_2) < \infty$. In Proposition 2.6 of [2] Borodachov, Hardin and Saff show that if A_1 belongs to a certain class of self-similar d-dimensional fractals, then $0 < \underline{g}_{s,d}(A_1) < \overline{g}_{s,d}(A_1) < \infty$. The set A_1 belongs to this class if

$$A_1 = \bigcup_{i=1}^{K} \varphi_i(A_1), \tag{1}$$

where each $\varphi_i : \mathbb{R}^p \to \mathbb{R}^p$ is a similitude with scaling L and where $\varphi_i(A_1) \cap \varphi_j(A_1) = \emptyset$ for all $i \neq j$. It is a consequence of results by Moran presented in [10] that there is a unique compact set A_1 that satisfies the condition in Equation (1), that the Hausdorff dimension of A_1 denoted by d is the solution of the equation $1 = KL^d$ and that $0 < \mathcal{H}^d(A_1) < \infty$. Cantor sets are a prominent example from this class.

For a concrete example consider the set that is the union of

$$A_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [3,4] \text{ and } x_2 = 0\}$$

and the set A_1 satisfying

$$A_1 = \bigcup_{i=1}^{4} \varphi_i(A_1),$$

where

$$\varphi_1(x) = \frac{x}{4}, \qquad \varphi_2(x) = \frac{x}{4} + \left(\frac{3}{4}, 0\right), \qquad \varphi_3(x) = \frac{x}{4} + \left(0, \frac{3}{4}\right) \qquad \text{and} \qquad \varphi_4(x) = \frac{x}{4} + \left(\frac{3}{4}, \frac{3}{4}\right).$$

The dimension of both A_1 and A_2 is 1, and both have positive and finite $\mathcal{H}^1$ measure.

2.2. **A weak-star cluster point of $\{\mu^{s,N}\}_{N=2}^{\infty} \subset \mathcal{M}_1(A)$.**

Here we shall identify a cluster point in the weak-star topology on $\mathcal{M}_1(A)$ of the sequence of measures $\{\mu^{s,N}\}_{N=2}^{\infty}$. This cluster point is one in which the s-energy of the sequence of configurations $\{\omega_N^s \cap A_1\}_{N=2}^{\infty}$ achieves $\underline{g}_{s,d}(A_1)$. We proceed by drawing on ideas and techniques developed by Hardin and Saff in [7].

An upper bound for $\dfrac{\mathcal{E}_s(A,N)}{N^{1+s/d}}$ can be obtained as follows: Choose natural numbers M_1 and $M_2 = N - M_1$ and consider a configuration of points $\tilde{\omega}_N$ such that $\#\tilde{\omega}_N \cap A_1 = M_1$ and $\#\tilde{\omega}_N \cap A_2 = M_2$, and where $E_s(\tilde{\omega}_N \cap A_1) = \mathcal{E}_s(A_1, M_1)$ and $E_s(\tilde{\omega}_N \cap A_2) = \mathcal{E}_s(A_2, M_2)$. Here the # denotes the number of elements in the set following it.

Bounding from above the interaction energy of points on A_1 and points on A_2 by $d(A_1,A_2)^{-s}N^2$ and dividing by $N^{1+s/d}$ gives

$$\frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \le \frac{E_s(\tilde{\omega}_N)}{N^{1+s/d}} \le \left(\frac{M_1}{N}\right)^{1+s/d} \frac{\mathcal{E}_s\left(A_1,M_1\right)}{{M_1}^{1+s/d}} + \left(\frac{M_2}{N}\right)^{1+s/d} \frac{\mathcal{E}_s\left(A_2,M_2\right)}{{M_2}^{1+s/d}} + d(A_1,A_2)^{-s}\frac{N^2}{N^{1+s/d}}. \tag{2}$$

Let $\left\{\underline{N}_n^1\right\}_{n=1}^{\infty} \subset \mathbb{N}$ be an increasing sequence so that

$$\lim_{n\to\infty} \frac{\mathcal{E}_s\left(A_1,\underline{N}_n^1\right)}{{\underline{N}_n^1}^{1+s/d}} = \underline{g}_{s,d}(A_1).$$

For $\alpha \in (0,1)$ let $\{N_n^{\alpha}\}_{n=1}^{\infty} \subset \mathbb{N}$ be the sequence defined by $N_n^{\alpha} = \left\lfloor \frac{1}{\alpha}\underline{N}_n^1 \right\rfloor$. Applying the bound given in Inequality (2) where N is chosen to be N_n^{α}, M_1 is chosen to be $\underline{N}_n^1$ and M_2 is chosen to be $N_n^{\alpha} - \underline{N}_n^1$ gives

$$\frac{\mathcal{E}_s\left(A,N_n^{\alpha}\right)}{{N_n^{\alpha}}^{1+s/d}} \le \left(\frac{\underline{N}_n^1}{N_n^{\alpha}}\right)^{1+s/d} \frac{\mathcal{E}_s\left(A_1,\underline{N}_n^1\right)}{{\underline{N}_n^1}^{1+s/d}} + \left(\frac{N_n^{\alpha}-\underline{N}_n^1}{N_n^{\alpha}}\right)^{1+s/d} \frac{\mathcal{E}_s\left(A_2,N_n^{\alpha}-\underline{N}_n^1\right)}{\left(N_n^{\alpha}-\underline{N}_n^1\right)^{1+s/d}} + d(A_1,A_2){N_n^{\alpha}}^{1-s/d}$$

For every $\varepsilon > 0$ we may find an $N_0 = N_0(\varepsilon)$ sufficiently high so that for every $N_n^{\alpha} > N_0$

$$\frac{\mathcal{E}_s\left(A,N_n^{\alpha}\right)}{{N_n^{\alpha}}^{1+s/d}} \le \alpha^{1+s/d}\underline{g}_{s,d}(A_1) + (1-\alpha)^{1+s/d} g_{s,d}(A_2) + \varepsilon. \tag{3}$$

The unique value of α that minimizes the right hand side of Inequality (3) is

$$\alpha^* = \frac{g_{s,d}(A_2)^{d/s}}{\underline{g}_{s,d}(A_1)^{d/s} + g_{s,d}(A_2)^{d/s}}.$$

Define $\{N_n^{\alpha^*}\}_{n=1}^{\infty}$ by $N_n^{\alpha^*} = \left\lfloor \frac{1}{\alpha^*}\underline{N}_n^1 \right\rfloor$.

To obtain a lower bound for $\frac{\mathcal{E}_s(A,N)}{N^{1+s/d}}$ begin by defining the function $N_1 : \mathbb{N} \to \mathbb{N}$ by

$$N_1(N) = \min_{\omega_N^s \subset A} \#(\omega_N^s \cap A_1).$$

Because an N-point s-energy-minimizing configuration ω_N^s may not be unique, we take a minimum over all N-point s-energy-minimizing configurations to determine the value $N_1(N)$. We define $N_2(N)$ to be $N - N_1(N)$. Note that $N_2(N) \ge \min_{\omega_N^s \subset A} \#(\omega_N^s \cap A_2)$.

Given N and a minimal N-point configuration ω_N^s such that $N_1(N) = \#\omega_N^s \cap A_1$, we may bound from below $\mathcal{E}_s(A,N) = E_s(\omega_N^s)$ by discarding the interaction energy between points in $\omega_N^s \cap A_1$ and points in $\omega_N^s \cap A_2$ and then further replacing the points in $\omega_N^s \cap A_1$ with a minimal $N_1(N)$-point configuration in A_1 and replacing the points in $\omega_N^s \cap A_2$ with a minimal $N_2(N)$-point configuration in A_2. The bound is then

$$\frac{\mathcal{E}_s\left(A,N\right)}{N^{1+s/d}} \ge \left(\frac{N_1(N)}{N}\right)^{1+s/d} \frac{\mathcal{E}_s\left(A_1,N_1(N)\right)}{N_1(N)^{1+s/d}} + \left(\frac{N_2(N)}{N}\right)^{1+s/d} \frac{\mathcal{E}_s\left(A_2,N_2(N)\right)}{N_2(N)^{1+s/d}}. \tag{4}$$

We would like to improve the bound above by employing the asymptotic properties of $\dfrac{\mathcal{E}_s\left(A_1,N_1(N)\right)}{N_1(N)^{1+s/d}}$ and $\dfrac{\mathcal{E}_s\left(A_1,N_1(N)\right)}{N_1(N)^{1+s/d}}$. This will require ensuring that both $N_1(N)$ and $N_2(N)$ tend to infinity as N tends to infinity. This will be accomplished by applications of Lemma 2.1, which employs ideas presented by Björck in [1].

Lemma 2.1. *Let B_1, $B_2 \subset \mathbb{R}^p$ be compact and of dimension $d > 0$. Further suppose* $\operatorname{diam}(B_2) < d(B_1,B_2)$*. Let $s > d$ and define*

$$\tilde{N}_1(N) = \min_{\omega_N^s \subset B_1 \cup B_2} \#(\omega_N^s \cap B_1),$$

where the minimum is taken over all N-point minimal s-energy configurations within $B_1 \cup B_2$. Then

$$\liminf_{N\to\infty} \tilde{N}_1(N) = \infty.$$

Proof. For sake of contradiction assume that

$$\liminf_{N\to\infty} \tilde{N}_1(N) = L < \infty,$$

then there is an increasing sequence $\{N_i\}_{i=1}^{\infty} \subset \mathbb{N}$ so that $\tilde{N}_1(N_i) = L$ for all $i \in \mathbb{N}$. The contradiction shall be that, for large enough N, a configuration that only places L points on B_1 cannot have minimal energy.

We first show that the quantity

$$R = \inf_{\substack{K \subset B_1 \\ \#K = L}} \max_{y \in B_1} d(y, K)$$

is positive. We may choose a sequence of L-point configurations in B_1, $\{K_n\}_{n=1}^{\infty}$, so that

$$\lim_{n\to\infty} \max_{y\in B_1} d(y, K_n) = R,$$

and because the L-fold product of B_1 with itself is compact, we may choose $\{K_n\}_{n=1}^{\infty}$ to be convergent to some K^*. Continuity of the function $B_1^L \ni K \to \max_{y\in B_1} d(y, K) \in \mathbb{R}$ allows us to conclude

$$\max_{y\in B_1} d(y, K^*) = R.$$

The set B_1 is of positive Hausdorff dimension so it is infinite, which is sufficient to conclude $R > 0$.

If $\tilde{N}_1(N_i) = L$ for all i, then for every i there is a point $r_i \in B_1$ that is separated from $\omega_{N_i}^s$ by at least R for any minimal N_i-point configuration $\omega_{N_i}^s \subset B_1 \cup B_2$. We may bound the potential energy at r_i due to the point in $\omega_{N_i}^s$ from above by

$$U(r_i) = \sum_{x\in\omega_{N_i}^s} \frac{1}{|r_j - x|^s} \le LR^{-s} + (N_i - L)d(B_1, B_2)^{-s},$$

where the first term is an upper bound for the contribution of the L points of $\omega_{N_i}^s \cap B_1$ and the second term is an upper bound for the $N_i - L$ points of $\omega_{N_i}^s \cap B_2$.

Alternatively, given any point $x_j \in \omega_{N_i}^s \cap B_2$ we may bound from below its point energy due to the other points by

$$U_j(x_j) = \sum_{x\in\omega_{N_i}^s \setminus\{x_j\}} \frac{1}{|x_j - x|^s} \ge (N_i - L - 1)\,\mathrm{diam}(B_2)^{-s}.$$

In this lower bound we have excluded the contribution to the point energy at x_j from the points in $\omega_{N_i}^s \cap B_1$, and bounded from below the contribution to the point energy at x_j from any point in $\omega_{N_i}^s \cap B_2\setminus\{x_j\}$ by $\mathrm{diam}(B_2)^{-s}$.

Because $d(B_1, B_2) > \mathrm{diam}(B_2)$, we may find N_i sufficiently high so that $U(r_i) < U_j(x_j)$, but then the energy of configuration $\omega_{N_i}^s$ could be reduced by moving the j^{th} point from x_j to r_j and this contradicts the assumption that $\omega_{N_i}^s$ has minimal energy. □

We may apply Lemma 2.1 to the sets under consideration by identifying either A_1 or A_2 as B_1. This is sufficient to show that

$$\liminf_{N\to\infty} N_1(N) = \liminf_{N\to\infty} N_2(N) = \infty.$$

Along the the sequence $\{N_n^{\alpha^*}\}_{n=1}^{\infty}$, we may apply the bound given in Inequality (3) and, because $N_1(N_n^{\alpha^*})$ and $N_1(N_n^{\alpha^*})$ grow to infinity, we may apply Inequality (4) giving, for $N_n^{\alpha^*}$ sufficiently high

$$\alpha^{*\,1+s/d}\underline{g}_{s,d}(A_1) + (1-\alpha^*)^{1+s/d} g_{s,d}(A_2) + 2\varepsilon \ge \left(\frac{N_1(N_n^{\alpha^*})}{N_n^{\alpha^*}}\right)^{1+s/d} \underline{g}_{s,d}(A_1) + \left(\frac{N_2(N_n^{\alpha^*})}{N_n^{\alpha^*}}\right)^{1+s/d} g_{s,d}(A_2).$$

Here we have bounded from below $\dfrac{\mathcal{E}_s\left(A_1, N_1(N_n^{\alpha^*})\right)}{N_1(N_n^{\alpha^*})^{1+s/d}}$ by $\underline{g}_{s,d}(A_1) - \varepsilon/2$ and bounded from below $\dfrac{\mathcal{E}_s\left(A_2, N_2(N_n^{\alpha^*})\right)}{N_2(N_n^{\alpha^*})^{1+s/d}}$ by $g_{s,d}(A_2) - \varepsilon/2$.

Because ε is arbitrary and because α^* is the unique minimizer of the left hand side of the above upper bound we may conclude that

$$\lim_{n\to\infty} \frac{N_1(N_n^{\alpha^*})}{N_n^{\alpha^*}} = \alpha^*.$$

If we let $\psi \in \mathcal{M}_1(A)$ be any weak-star cluster point of $\{\mu^{s,N_n^{\alpha^*}}\}_{n=1}^{\infty}$, and if we choose a continuous function $\phi : A \to \mathbb{R}$ such that $\phi(x) = 1$ for all $x \in A_1$ and $\phi(y) = 0$ for all $y \in A_2$, then

$$\int \phi d\psi = \psi(A_1) = \alpha^*.$$

This is sufficient to prove Lemma 2.2

Lemma 2.2. *Let A be the disjoint union of two compact sets A_1 and A_2 that meet the following conditions:*

1. *both A_1 and A_2 are of Hausdorff dimension d,*
2. $\operatorname{diam}(A_1)$ *and* $\operatorname{diam}(A_2)$ *are both less than $d(A_1, A_2)$,*
3. *for some $s > d$, $0 < \underline{g}_{s,d}(A_1) \le \overline{g}_{s,d}(A_1) < \infty$ and*
4. *for the same s, $g_{s,d}(A_2)$ exists and $0 < g_{s,d}(A_2) < \infty$.*

Then there is a weak-star cluster-point $\psi \in \mathcal{M}_1(A)$ of the sequence of measures $\{\mu^{s,N}\}_{N=2}^{\infty}$ so that

$$\psi(A_1) = \frac{g_{s,d}(A_2)}{\underline{g}_{s,d}(A_1) + g_{s,d}(A_2)}.$$

3. Rate of Change in $\dfrac{\mathcal{E}_s(K,N)}{N^{1+s/d}}$ for $K \subset \mathbb{R}^p$ Compact

In this section we consider a compact set $K \subset \mathbb{R}^p$ so that $0 < \overline{g}_{s,d}(K) < \infty$, and bound from below the rate of change of the function $G : \mathbb{N} \to \mathbb{R}$ defined by

$$G(N) = \frac{\mathcal{E}_s(K,N)}{N^{1+s/d}}.$$

This is the content of Lemma 3.1 In Corollary 3.2 we use Lemma 3.1 to show that, if for some N_0, $G(N_0)$ is close to $\overline{g}_{s,d}(K)$, then for some N' that is larger than N_0 and for which the ratio N_0/N' is close to 1, we will have that $G(N')$ is also close to $\overline{g}_{s,d}(K)$.

Lemma 3.1. *Let $K \subset \mathbb{R}^p$ be compact and $s > d = \dim K > 0$. Let $N \ge 2$ and $N' \ge 1$ be natural numbers and let $\kappa = \frac{N'}{N}$. Then*

$$G((1+\kappa)N) \ge G(N) - \left(1 + \frac{s}{d}\right)\kappa G(N).$$

Proof. From our assumptions

$$\begin{aligned} G((1+\kappa)N) = \frac{\mathcal{E}_s(K,N+N')}{(N+N')^{1+s/d}} &> \left(\tfrac{N}{N+N'}\right)^{1+s/d} \tfrac{\mathcal{E}_s(K,N)}{N^{1+s/d}} \\ &= \left(\tfrac{1}{1+\kappa}\right)^{1+s/d} G(N) \\ &\ge \left(1 - \left(1 + \tfrac{s}{d}\right)\kappa\right) G(N) \\ &= G(N) - \left(1 + \tfrac{s}{d}\right)\kappa G(N), \end{aligned}$$

where the last inequality follows from a first-order expansion of $h(\kappa) := \left(\frac{1}{1+\kappa}\right)^{1+s/d}$ about zero and the convexity of h. □

Corollary 3.2. *Let $K \subset \mathbb{R}^p$ be compact and $s, d > 0$ so that $0 < \overline{g}_{s,d}(K) < \infty$. Let $\{M_n\}_{n=1}^{\infty} \subset \mathbb{N}$ be an increasing sequence so that*

$$\lim_{n\to\infty} \frac{\mathcal{E}_s(K,M_n)}{{M_n}^{1+s/d}} = \overline{g}_{s,d}(K).$$

For every $\varepsilon > 0$ there is a $\delta > 0$ and an $M_0 < \infty$ so that if, for some $M' \in \mathbb{N}$ and $\tilde{M} \in \{M_n\}_{n=1}^{\infty}$,

$$M_0 < \tilde{M} < M' < \frac{\tilde{M}}{1-\delta},$$

then

$$\frac{\mathcal{E}_s(K,M')}{{M'}^{1+s/d}} \ge \overline{g}_{s,d}(K) - \varepsilon.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Choose M_0 so that for any $\tilde{M} \in \{M_n\}_{n=1}^{\infty}$ greater than M_0

$$\frac{\mathcal{E}_s\left(K, \tilde{M}\right)}{\tilde{M}^{1+s/d}} > \overline{g}_{s,d}(K) - \frac{\varepsilon}{2} \tag{5}$$

Choose κ_0 sufficiently small so that for all $\kappa \in (0, \kappa_0)$,

$$(1 + s/d)\kappa \sup_{M>M_0} \frac{\mathcal{E}_s\left(K, M\right)}{M^{1+s/d}} < \frac{\varepsilon}{2}. \tag{6}$$

Combining Lemma 3.1 and Inequalities (5) and (6) gives for any $\tilde{M} \in \mathbb{N}$ and $\kappa \in (0, \kappa_0)$ where $(1 + \kappa)\tilde{M} \in \mathbb{N}$

$$\frac{\mathcal{E}_s\left(K, (1+\kappa)\tilde{M}\right)}{\left((1+\kappa)\tilde{M}\right)^{1+s/d}} \geq \frac{\mathcal{E}_s\left(K, \tilde{M}\right)}{\tilde{M}^{1+s/d}} - (1 + s/d)\kappa \frac{\mathcal{E}_s\left(K, \tilde{M}\right)}{\tilde{M}^{1+s/d}} \geq \overline{g}_{s,d}(K) - \varepsilon. \tag{7}$$

Choose $\delta = \dfrac{\kappa_0}{1 + \kappa_0}$. If M' and $\tilde{M}$ are such that $M_0 < \tilde{M} < M' < \frac{\tilde{M}}{1-\delta}$, and if $M' = (1 + \kappa)\tilde{M}$, then $\kappa < \kappa_0$ and the bound in Inequality (7) ensures

$$\frac{\mathcal{E}_s\left(K, M'\right)}{M'^{1+s/d}} \geq \overline{g}_{s,d}(K) - \varepsilon.$$

□

4. Non-Convergence of $\{\mu^{s,N}\}_{N=2}^{\infty}$ in the Weak-Star Topology

In this section we prove Theorem 1.1 – that the sequence of measures $\{\mu^{s,N}\}_{N=2}^{\infty} \subset \mathcal{M}_1(A)$ cannot converge in the weak-star topology on $\mathcal{M}_1(A)$. If the sequence did converge, it must converge to the measure ψ identified in Lemma 2.2, this will be shown to lead to a contradiction.

Proof of Theorem 1.1. For sake of contradiction, assume that the sequence $\{\mu^{s,N}\}_{N=2}^{\infty} \subset \mathcal{M}_1(A)$ converged in the weak-star topology on $\mathcal{M}_1(A)$. Since this topology separates elements of $\mathcal{M}_1(A)$ and since, by Lemma 2.1, a subsequence of $\{\mu^{s,N}\}_{N=2}^{\infty}$ converges to ψ, the assumption would require that $\mu^{s,N}$ converges to ψ in the weak-star sense as $N \to \infty$ implying

$$\lim_{N\to\infty} \frac{N_1(N)}{N} = \alpha^* = \frac{g_{s,d}(A_2)}{\underline{g}_{s,d}(A_1) + g_{s,d}(A_2)} \tag{8}$$

Let $\left\{\overline{N}_n^1\right\}_{n=1}^{\infty} \subset \mathbb{N}$ be an increasing sequence so that

$$\lim_{n\to\infty} \frac{\mathcal{E}_s\left(A_1, \overline{N}_n^1\right)}{\overline{N}_n^{1\,1+s/d}} = \overline{g}_{s,d}(A_1).$$

Our intention is to find $N''' \in \left\{\overline{N}_n^1\right\}_{n=1}^{\infty}$ and an $N' \in \mathbb{N}$ so that we may apply Corollary 3.2 where $\tilde{M}$ is identified with N''' and M' is identified with $N_1(N')$.

We proceed as follows: Let $\varepsilon > 0$. Let δ and M_0 be as provided by Corollary 3.2 applied to the case $K = A_1$ and $\{M_n\}_{n=1}^{\infty} = \left\{\overline{N}_n^1\right\}_{n=1}^{\infty}$. One may verify that it is possible to choose $\gamma \in (0, 1)$ such that, for $\gamma' = \dfrac{2\gamma}{1-\gamma}$, the following (motivated by inequalities arising later in the proof) hold:

$$\gamma' + (1 + \gamma')\left(1 - \frac{1}{1 + 3\gamma}\right) < \delta \qquad \text{and} \qquad \frac{1 + \frac{5}{2}\gamma}{1 + \gamma'} \geq 1.$$

We choose such a γ. Define the non-decreasing sequence $\{\tilde{N}_n\}_{n=1}^{\infty}$ by the equation $\tilde{N}_n = \left\lfloor (1 + 3\gamma)\overline{N}_n^1 \right\rfloor$.

From Equation (8) and from Lemma 2.2, we may further increase M_0 so that the following all hold

$$\sup_{N\geq M_0} \left|\frac{N_1(N)}{N} - \alpha^*\right| < \min\{\gamma\alpha^*, \varepsilon\}, \tag{9}$$

$$\sup_{N\geq M_0} \frac{\mathcal{E}_s\left(A_1, N_1(N)\right)}{N_1(N)^{1+s/d}} \leq \overline{g}_{s,d}(A_1) + \varepsilon \qquad \text{and} \tag{10}$$

$$\sup_{N\geq M_0} \left| \frac{\mathcal{E}_s\left(A_2, N_2(N)\right)}{N_2(N)^{1+s/d}} - g_{s,d}(A_2) \right| < \varepsilon. \tag{11}$$

$$\left(1 + \frac{5}{2}\gamma\right) N < \lfloor (1+3\gamma)N \rfloor \qquad \text{for all } N > M_0.$$

Inequality (9) implies

$$\sup_{N\geq M_0} \left| \frac{N_1(N)}{\alpha^* N} - 1 \right| < \gamma. \tag{12}$$

Choose $N' \in \mathbb{N}$ and $N'' \in \{\tilde{N}_n\}_{n=1}^{\infty}$ so that both are greater than M_0, and so that

$$\left| 1 - \frac{N''}{\alpha^* N'} \right| < \gamma. \tag{13}$$

This allows the following bound

$$\begin{aligned} \left|1 - \frac{N''}{N_1(N')}\right| &= \left|1 - \tfrac{N''}{\alpha^* N'} + \tfrac{N''}{\alpha^* N'} - \tfrac{N''}{N_1(N')}\right| \\ &\leq \left|1 - \tfrac{N''}{\alpha^* N'}\right| + \left|\tfrac{N''}{\alpha^* N'} - \tfrac{N''}{N_1(N')}\right| \\ &\leq \gamma + \left(\tfrac{N''}{\alpha^* N'}\right)\left|1 - \tfrac{\alpha^* N'}{N_1(N')}\right| \\ &\leq \gamma + (1+\gamma)\tfrac{\gamma}{1-\gamma} \\ &= \tfrac{2\gamma}{1-\gamma} = \gamma'. \end{aligned} \tag{14}$$

The last inequality above follows from Inequalities (12) and (13). Now let N''' denote the element of $\left\{\overline{N}_n^1\right\}_{n=1}^{\infty}$ so that $N'' = \lfloor (1+3\gamma)N''' \rfloor$. Then

$$\begin{aligned} \left|1 - \frac{N'''}{N_1(N')}\right| &= \left|1 - \tfrac{N''}{N_1(N')} + \tfrac{N''}{N_1(N')} - \tfrac{N'''}{N_1(N')}\right| \\ &\leq \left|1 - \tfrac{N''}{N_1(N')}\right| + \left|\tfrac{N''}{N_1(N')} - \tfrac{N'''}{N_1(N')}\right| \\ &\leq \gamma' + \tfrac{N''}{N_1(N')}\left|1 - \tfrac{N'''}{N''}\right| \\ &\leq \gamma' + (1+\gamma')\left|1 - \tfrac{N'''}{\lfloor 1+3\gamma \rfloor N'''}\right| \\ &\leq \gamma' + (1+\gamma')\left(1 - \tfrac{1}{1+3\gamma}\right). \end{aligned}$$

Here the second to last inequality follows from Inequality (14). Inequality (14) also implies

$$1 - \frac{N''}{N_1(N')} \geq -\gamma', \quad \text{hence} \quad N_1(N') \geq \frac{N''}{\gamma'+1} = \frac{\lfloor (1+3\gamma)N''' \rfloor}{\gamma'+1} \geq N''' \frac{\left(1 + \frac{5}{2}\gamma\right)}{\gamma'+1}$$

Because γ was chosen to ensure that

$$\frac{\left(1 + \frac{5}{2}\gamma\right)}{\gamma'+1} \geq 1, \qquad \text{and} \qquad \gamma' + (1+\gamma')\left(1 - \frac{1}{1+3\gamma}\right) < \delta$$

we have that $N_1(N') \geq N'''$ and

$$1 - \frac{N'''}{N_1(N')} < \delta.$$

With this we may employ Corollary 3.2 where $\tilde{M}$ is identified with N''' and M' is identified with $N_1(N')$, giving the bound

$$\frac{\mathcal{E}_s\left(A_1, N_1(N')\right)}{N_1(N')^{1+s/d}} \geq \overline{g}_{s,d}(A_1) - \varepsilon.$$

This, combined with Inequalities (4) and (11), gives

$$\frac{\mathcal{E}_s\,(A,N')}{N'^{1+s/d}} \geq \left(\frac{N_1(N')}{N'}\right)^{1+s/d} (\overline{g}_{s,d}(A_1)-\varepsilon) + \left(N' - \frac{N_1(N')}{N'}\right)^{1+s/d} (g_{s,d}(A_2)-\varepsilon).$$

In light of Inequality (9) we have

$$\frac{\mathcal{E}_s\,(A,N')}{N'^{1+s/d}} \geq (\alpha^*-\varepsilon))^{1+s/d}(\overline{g}_{s,d}(A_1)-\varepsilon) + (1-\alpha^*-\varepsilon)^{1+s/d}(g_{s,d}(A_2)-\varepsilon).$$

In the preceding arguments, ε was an arbitrary positive number that constrained N' to be above some value. We may then construct a sequence $\{N'_n\}_{n=1}^{\infty}$ of such N' so that

$$\lim_{n\to\infty} \frac{\mathcal{E}_s\,(A,N'_n)}{N_n'^{1+s/d}} \geq \alpha^{*1+s/d}\overline{g}_{s,d}(A_1) + (1-\alpha^*)^{1+s/d} g_{s,d}(A_2).$$

Now, for each $N'_n \in \{N'_n\}_{n=1}^{\infty}$ that is larger than $2M_0$, and for any two natural numbers M_n^1 and $M_n^2 = N'_n - M_n^1$ both greater than M_0, Inequalities (2), (10) and (11) give

$$\frac{\mathcal{E}_s\,(A,N'_n)}{N_n'^{1+s/d}} \leq \left(\frac{M_n^1}{N'_n}\right)^{1+s/d} (\overline{g}_{s,d}(A_1)+\varepsilon) + \left(\frac{M_n^2}{N'_n}\right)^{1+s/d} (g_{s,d}(A_2)+\varepsilon) + N_n'^{1-s/d} d(A_1,A_2).$$

For an arbitrary $\beta \in (0,1)$ we may choose M_n^1 and M_n^2 so that $\frac{M_n^1}{N'_n} \to \beta$ and $\frac{M_n^2}{N'_n} \to 1-\beta$ as $n \to \infty$. Note also that, because N'_n goes to infinity as n goes to infinity, the term $N_n'^{1-s/d} d(A_1,A_2)$ goes to zero as n grows. For any β we obtain the following bound

$$\beta^{1+s/d}\overline{g}_{s,d}(A_1) + (1-\beta)^{1+s/d} g_{s,d}(A_2) \geq \lim_{n\to\infty} \frac{\mathcal{E}_s\,(A,N'_n)}{N_n'^{1+s/d}} \geq \alpha^{*1+s/d}\overline{g}_{s,d}(A_1) + (1-\alpha^*)^{1+s/d} g_{s,d}(A_2). \tag{15}$$

The function $f(\beta) = \beta^{1+s/d}\overline{g}_{s,d}(A_1) + (1-\beta)^{1+s/d} g_{s,d}(A_2)$ is uniquely minimized by

$$\beta = \frac{g_{s,d}(A_2)}{\overline{g}_{s,d}(A_1) + g_{s,d}(A_2)} \neq \frac{g_{s,d}(A_2)}{\underline{g}_{s,d}(A_1) + g_{s,d}(A_2)} = \alpha^*,$$

invalidating Inequality (15). This is sufficient to show that $\{\mu^{s,N}\}_{N=2}^{\infty}$ cannot converge in the weak-star topology on $\mathcal{M}_1(A)$. ◻

References

[1] G. Björck. Distributions of positive mass, which maximize a certain generalized energy integral. *Ark. Mat.*, 3:255 – 269, 1956.

[2] S. V. Borodachov, D. P. Hardin, and E. B. Saff. Asymptotics of best-packing on rectifiable sets. *Proc. Amer. Math. Soc.*, 135(8):2369–2380 (electronic), 2007.

[3] S. V. Borodachov, D. P. Hardin, and E. B. Saff. Asymptotics for discrete weighted minimal energy problems on rectifiable sets. *Trans. Amer. Math. Soc.*, 360(3):1559–1580, March 2008.

[4] M. T. Calef. Riesz s-equilibrium measures on d-dimensional fractal sets as s approaches d. *Journal of Mathematical Analysis and Applications*, In Press, Corrected Proof:–, 2010.

[5] M. T. Calef and D. P. Hardin. Riesz *s*-equilibrium measures on *d*-rectifiable sets as *s* approaches *d*. *Potential Analysis*, 30(4):385–401, May 2009.

[6] B. Gustafsson and M. Putinar. The exponential transform: a renormalized Riesz potential at critical exponent. *Indiana Univ. Math. J.*, 52(3):527–568, 2003.

[7] D. P. Hardin and E. B. Saff. Minimal Riesz energy point configurations for rectifiable *d*-dimensional manifolds. *Adv. Math.*, 193(1):174–204, 2005.

[8] N. S. Landkof. *Foundations of Modern Potential Theory*. Springer-Verlag, New York, 1973.

[9] P. Mattila. *Geometry of Sets and Measures in Euclidian Spaces*. Cambridge University Press, Cambridge, UK, 1995.

[10] P. A. P. Moran. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.*, 42:15–23, 1946.

[11] M. Putinar. A renormalized Riesz potential and applications. In *Advances in constructive approximation: Vanderbilt 2003*, Mod. Methods Math., pages 433–465. Nashboro Press, Brentwood, TN, 2004.

M. T. Calef: Computational Physics (CCS-2), Los Alamos National Laboratory, Los Alamos, NM 87545, USA
E-mail address: mcalef@lanl.gov